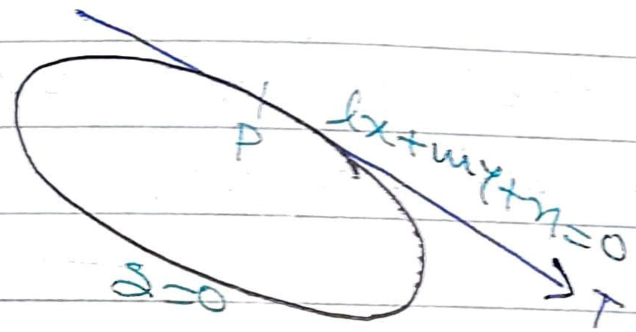
CONDITION OF TANGENCY.

THEOREM:

To find the condition that the straight line $lx + my + n$ may touch the conic $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Proof:



given eq^n of conic be

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{--- (1)}$$

given eq^n of st. line is

$$lx + my + n = 0 \quad \text{--- (2)}.$$

$$y = \frac{-(lx+n)}{m}$$

putting the value of y in eq^n (1) we

$$ax^2 + 2hx\left\{-\frac{(lx+n)}{m}\right\} + b\left\{-\frac{(lx+n)}{m}\right\}^2 + 2gx + 2f\left\{-\frac{(lx+n)}{m}\right\} + c = 0$$

or, $$ax^2 - \frac{2hx(lx+n)}{m} + \frac{b(lx+n)^2}{m^2} + 2gx - \frac{2f(lx+n)}{m} + c = 0.$$

or, $$\frac{am^2x^2 - 2hmx(lx+n) + b(lx+n)^2 + 2gm^2x - 2fm(lx+n) + cm^2}{m^2} = 0$$

or, $am^2x^2 - 2hmlx^2 - 2hmnx + b(l^2x^2 + n^2 + 2lxn) + 2gm^2x - 2flmx - 2fnm + cm^2 = 0$

or, $(am^2 - 2hml + bl^2)x^2 + \{-2hmn + 2bln + 2gm^2 - 2flm\}x + (bn^2 - 2fnm + cm^2) = 0$

or, $(am^2 - 2hml + bl^2)x^2 + 2\{bln - hmn - flm + gm^2\}x + (bn^2 - 2fnm + cm^2) = 0.$ ——— (3)

This is a quadratic equation in x, so it has 2 roots. If the st. line (ii) will be tangent to the given conic then it will touch it at only one point. So, roots of eq^n (iii) must be equal.

The condition for which roots of eq^n (3) be equal is that its discriminant i.e. $D = 0$

$\{2(bln - hmn - flm + gm^2)\}^2 - 4(am^2 - 2hlm + bl^2)(bn^2 - 2fmn + cm^2) = 0$

or, $4(bln - hmn + flm + gm^2)^2 = 4(am^2 \mp 2hlm + bl^2)(bn^2 - 2fmn + cm^2)$

$\therefore$ $(bln - hmn + flm + gm^2)^2 = (am^2 - 2hlm + bl^2)(bn^2 - 2fmn + cm^2)$

which is the required condition.

PARTICULAR CASES.

CASE (A)

To find the condition that the st line $y = mx + c$ may touch the parabola $y^2 = 4ax$.

Proof:

given eqn of parabola is

$y^2 = 4ax$ ——— (1)

and the st. line is

$y = mx + c$ ——— (2).

putting $y = mx + c$ in eqn (1) we get

$(mx + c)^2 = 4ax$.

or, $m^2x^2 + 2mxc + c^2 = 4ax$

$m^2x^2 + 2mxc + c^2 - 4ax = 0$

or, $m^2x^2 + (2mc - 4a)x + c^2 = 0$

or, $m^2x^2 + 2(mc - 2a)x + c^2 = 0$. ——— (3)

This is quadratic eqn in x, so it has 2 roots.
If the roots of eqn (3) are unequal then we get 2 different point of x coordinate of 2 different point of intersection of a straight line (2) with parabola (1)

In order to st. line (2) touch the parabola (1) then the roots of eqn (3) must be equal
The condition for which roots of eqn (3) be equal is that its discriminant be equal to zero.

i.e. $D = 0$.

or, $\{2(mc - 2a)\}^2 - 4.m^2.c^2 = 0$.

$4(mc - 2a)^2 = 4m^2c^2$.

or, $m^2c^2 + 4a^2 - 4amc = 4m^2c^2$.

$4a^2 = 4amc$

$a = mc$.

$$\boxed{c = \frac{a}{m}}$$

which is the required condition.

NOTE:

The equation of tangent to the parabola $y^2 = 4ax$ in terms of slope is $\boxed{y = mx + \frac{a}{m}}$

Case (B)

To find the condition that the st. line $y = mx + c$ may touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Proof:

given equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{--- (1)}.$$

& the st. line is

$$y = mx + c \quad \text{--- (2)}$$

we know that the eqn of tangent to the ellipse at any point $(a\cos\phi, b\sin\phi)$ is

$$\frac{x\cos\phi}{a} + \frac{y}{b}\sin\phi = 1 \quad \text{--- (3)}.$$

$$\frac{y}{b}\sin\phi = 1 - \frac{x}{a}\cos\phi.$$

$$\frac{y}{\frac{b}{\sin\phi}} = \left(-\frac{\cos\phi}{a}\right)x + 1$$

If the st. line (2) will be tangent to the ellipse (1) ~~then~~ at a point for which the st. line will be tangent then eq^n (2) & (3) must be identical.

On compairing.

$$\frac{1}{\sin\phi/b} = \frac{m}{-\frac{\cos\phi}{a}} = \frac{c}{1}$$

$$\frac{b/\sin\phi}{} = -\frac{am}{+\cos\phi} = \frac{c}{1}$$

$$\frac{b}{\sin\phi} = -\frac{am}{\cos\phi} = \frac{c}{1} \quad \text{—— (4)}$$

From (4).

$$c = \frac{b}{\sin\phi} \Rightarrow b = c\sin\phi \quad \text{—— (5)}$$

$$\& \quad c = -\frac{am}{\cos\phi} \Rightarrow c\cos\phi = -am \quad \text{—— (6)}$$

squaring and adding (5) & (6).

$$b^2 + a^2m^2 = c^2(\sin^2\phi + \cos^2\phi)$$

$$\therefore \boxed{c = \pm\sqrt{b^2 + a^2m^2}}$$

which is the required cond^n.

NOTE:

The eqⁿ of tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in terms of slope is

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

CASE (c)

To find the condition that the st. line $y = mx + c$ may touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Proof:

given eqⁿ of conic is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{--- (1)}$$

& the st. line is

$$y = mx + c \quad \text{--- (2)}$$

we know that the eqⁿ of tangent to the hyperbola at any point $(a\sec\phi, b\tan\phi)$ is

$$\frac{x\sec\phi}{a} - \frac{y\tan\phi}{b} = 1$$

$$\frac{y\tan\phi}{b} = \frac{x\sec\phi}{a} - 1$$

or,

$$\frac{y}{\frac{b}{\tan\phi}} = \frac{x}{\frac{a}{\sec\phi}} - 1 \quad \text{--- (3)}$$

If the st. line ② and ③ will be tangent to the ellipse hyperbola at a point for which the st. line will be tangent then eqⁿ ② and ③ must be identical

On comparing.

$$\frac{1}{\frac{\tan\phi}{b}} = \frac{am}{\frac{\sec\phi}{a}} = -c \quad \text{——} (4)$$

$$c = \frac{-b}{\tan\phi} \Rightarrow b = -c\tan\phi \quad \text{——} (5)$$

$$\& \quad c = \frac{-am}{\sec\phi} \Rightarrow am = -c\sec\phi \quad \text{——} (6).$$

Squaring and subtracting (5) and (6) we get

$$b^2 - a^2m^2 = c^2(\tan^2\phi - \sec^2\phi)$$

or, $a^2m^2 - b^2 = c^2(\sec^2\phi - \tan^2\phi)$

or, $a^2m^2 - b^2 = c^2.$

$$\therefore \quad \boxed{c = \pm\sqrt{a^2m^2 - b^2}}$$

which is the required condition.

NOTE:

The equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in terms of slope is

$$\boxed{y = mx \pm \sqrt{a^2m^2 - b^2}}$$